

# On Amplification by Weak Measurement

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We analyze the amplification by the Aharonov-Albert-Vaidman weak quantum measurement on a Sagnac interferometer [P. B. Dixon et al., Phys. Rev. Lett. 102, 173601 (2009)] up to all orders of the coupling strength between the measured system and the measuring device. The amplifier transforms a small tilt of a mirror into a large transverse displacement of the laser beam. The conventional analysis has shown that the measured value is proportional to the weak value, so that the amplification can be made arbitrarily large in the cost of decreasing output laser intensity. It is shown that the measured displacement and the amplification factor are in fact not proportional to the weak value and rather vanish in the limit of infinitesimal output intensity. We derive the optimal overlap of the pre- and post-selected states with which the amplification become maximum. We also show that the nonlinear effects begin to arise in the performed experiments so that any improvements in the experiment, typically with an amplification greater than 100, should require the nonlinear theory in translating the observed value to the original displacement.

*Introduction.*— The standard theory of measurement in quantum mechanics deals with the situation that one performs a measurement on a quantum state to obtain a measured value and the resulting state according to certain probabilistic laws. It was established by von Neumann [1] in the case of projective measurements and generalized later to non-projective measurements [2–4]. In experiments as well as in theory, weak measurements, where the system is weakly coupled with, hence weakly disturbed by, the measuring device, have been widely considered and have proved to be useful.

Aharonov, Albert, and Vaidman (AAV) [5] proposed a particular type of weak measurement which is characterized by the pre- and post-selection (PPS) of the system. One prepares the initial state  $|i\rangle$  of the system and that  $|\Phi_i\rangle$  of the device, and after a certain interaction between the system and the meter, one post-selects a state  $|f\rangle$  of the system and reads the meter value. If one measures an observable  $A$  of the system, one obtains the *weak value*

$$A_w := \frac{\langle f|A|i\rangle}{\langle f|i\rangle}. \quad (1)$$

A peculiar feature of the weak value is that it can take “strange” values which are outside the range of the eigenvalues of  $A$  and may even be complex. One can easily see from (1) that when  $|i\rangle$  and  $|f\rangle$  are almost orthogonal, the absolute value of the weak value can be arbitrarily large. This also means that when the weak value  $A_w$  becomes very large, the probability  $|\langle f|i\rangle|^2$  of successful post-selection of the state  $|f\rangle$  becomes very small. For example, Aharonov et al. [5] has theoretically demonstrated that a spin component of a spin-1/2 particle can have a weak value of 100. Related phenomena have been confirmed in quantum optics [6]. There are attempts to understand the meaning of  $A_w$  by formulating a general theory [7, 8] and by examining negative probability and strange weak values [9–11].

From these features, we may say that *the weak measurement allows one to trade the probability of successful*

*post-selection for the magnitude of a physical quantity.* Recently, the weak measurement were applied to precision measurements, which took advantages of the “trade” above. Hosten and Kwiat [12] measured the spin Hall effect of light by amplifying the small displacement of the light ray by weak measurement. Dixon et al. [13] were able to measure a small tilt of 400frad of a mirror in a Sagnac interferometer. There the observed values are proportional to the weak value (1) of some physical quantity  $A$ . Therefore the amplification can be arbitrarily large in principle. The conventional analysis of the weak measurement, as will be reviewed shortly, assumes a linear approximation with respect to the coupling between the system and the meter. Thus it is worth pursuing the true behavior of the amplification factor in the full analysis. Recently, Wu and Li [14] extended the general formulation by Jozsa [7] and gave a formal power expansion analysis of the weak measurement. They also discussed its effect with a simple example.

In this paper, we perform a direct analysis of the amplification by the weak measurement on the Sagnac interferometer, up to all orders of the coupling between the system and the meter. In particular, it is shown that the measured value as well as the amplification factor are bounded in the limit that the PPS states become orthogonal. One therefore *cannot* trade the probability of successful post-selection for the amplification as much as one wants. We also show that in the performed experiments the nonlinear behavior starts to appear so that improvements of the amplifier will require the nonlinear analysis.

*Amplification by Sagnac interferometer.*— The precision measurement by Sagnac interferometer [13] makes use of the weak measurement where the which-path degree of freedom is the measured system and the transverse displacement of the laser is the measuring device (meter). The input laser is divided by the beam splitter to the clockwise path  $|\odot\rangle$  and the anticlockwise one

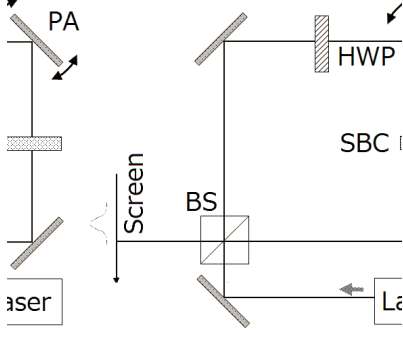


FIG. 1: Amplifier by the Sagnac interferometer. The input laser beam is divided by the beam splitter (BS) and come back again after given a transverse momentum and a phase difference. The momentum is given by a tilted mirror controlled by a piezo-actuator (PA). The phase difference is given by the SBC under a polarization control. One measures the beam position at the “screen,” which is usually a CCD detector.

$|\odot\rangle$  (Figure 1). The beams go through the beam splitter again and all photons go out to the port where the input beam comes in (bright port). The other port is called the dark port. If there is a phase difference between the clockwise and anticlockwise beams, there are photons coming to the dark port. This is done in the following manner. First, the input beam is polarized horizontally. Second, the polarization is changed to vertical by the half-wave plate either before or after the beam passes the Soleil-Babinet compensator (SBC) depending on which path the photon takes. Third, the SBC gives a phase difference  $\phi$  between the horizontally and vertically polarized beams. Finally, one measures the transverse displacement in the dark port and obtain its expectation value  $\langle x \rangle$ . This method gives an amplification of the small tilt of the mirror.

The initial state of the which-path degrees of freedom is given by  $|i\rangle = \frac{1}{\sqrt{2}}(e^{-i\phi/2}|\odot\rangle + ie^{i\phi/2}|\oslash\rangle)$ . When  $\phi = 0$ , the dark port is completely dark. The initial state for the transverse displacement is  $|\Phi_i\rangle = \int dx |x\rangle \Phi_i(x)$ . Thus the initial state of the whole system is given by

$$|\Psi_i\rangle = |\Phi_i\rangle |i\rangle. \quad (2)$$

The tilt of the mirror gives a transverse momentum  $\pm k$ , where the signature depends on the path of the beam, so that the effect is described by a unitary operator  $e^{-ikxA}$ , where  $A := |\odot\rangle\langle\odot| - |\oslash\rangle\langle\oslash|$  is the operator which distinguishes the paths of the beam.

The post-selected state is the dark port state  $|f\rangle = \frac{1}{\sqrt{2}}(|\odot\rangle - |\oslash\rangle)$ . Then the final state of the meter is given by  $|\Phi_f\rangle = \langle f|\Psi\rangle = \int dx |x\rangle \Phi_i(x) \langle f|e^{-ikxA}|i\rangle$ . The standard argument of the weak measurement is to expand  $\langle f|e^{-ikxA}|i\rangle$  and write  $\langle f|A|i\rangle$  by the weak value  $A_w$  and regroup those to an exponential function,  $\langle f|e^{-ikxA}|i\rangle \simeq$

$\langle f|i\rangle(1 - ikxA_w) \simeq \langle f|i\rangle e^{-ikxA_w}$ . One thereby has

$$|\Phi_f\rangle \simeq \langle f|i\rangle \int dx |x\rangle \Phi_i(x) e^{-ikxA_w}. \quad (3)$$

This gives the final read of the meter,

$$\langle x \rangle = \frac{\langle \Phi_f | x | \Phi_f \rangle}{\langle \Phi_f | \Phi_f \rangle} \simeq 2ka^2 \text{Im} A_w = 2ka^2 \cot \frac{\phi}{2}, \quad (4)$$

where the wave function  $\Phi_i(x)$  is assumed to be even, and  $A_w = i \cot \frac{\phi}{2}$  has been used. The length  $a = \sqrt{\langle x^2 \rangle_i}$  is the input beam radius, where  $\langle \bullet \rangle_i$  denotes the expected value with respect to the initial state. In the case of diverging paraxial beam performed in the experiments [13], the result (4) is modified as

$$\langle x \rangle = 2k\sigma[(1 - \gamma)a + \gamma\sigma] \text{Im} A_w, \quad (5)$$

where  $\gamma := l_{\text{lm}}/(l_{\text{lm}} + l_{\text{md}})$  with  $l_{\text{lm}}$  and  $l_{\text{md}}$  being the path lengths between the lens and the mirror, and between the mirror and the detector, respectively, and  $\sigma$  is the radius of the beam at the detector. Note that (5) is consistent with [7]; in particular, the imaginary part of a weak value gives a shift in  $x$  when the coupling Hamiltonian is proportional to  $xA$ .

Dixon et al. defined the amplification factor

$$\mathcal{A} = \frac{|\langle x \rangle|}{\delta}, \quad (6)$$

as the ratio of the displacement  $\langle x \rangle$  compared with that  $\delta$  without interferometer, where  $\delta = kl_{\text{md}}/k_0$  with  $l_{\text{md}}$  being the length of the path from the mirror to the detector and  $k_0$  being the wavelength of the laser. Eq. (5) shows that the displacement  $\langle x \rangle$  as well as the amplification factor  $\mathcal{A}$  are proportional to the weak value (1). This implies that those quantities diverges in the limit  $\phi \rightarrow 0$  when the initial and final states become orthogonal. Though in this limit the output power, which is proportional to  $|\langle f|i\rangle|^2 = \sin^2 \frac{\phi}{2}$ , become infinitesimally small and it becomes extremely difficult to perform an experiment, but apart from that, the displacement  $\langle x \rangle$  and the amplification  $\mathcal{A}$  can in principle become arbitrarily large.

In the rest of the paper, we shall calculate how the measurement result in the weak measurement differs from the weak value and show that  $\langle x \rangle$  and  $\mathcal{A}$  actually do not diverge even in the limit  $\langle f|i\rangle \rightarrow 0$ .

*Nonlinear effect.*— Let us derive the amplified displacement  $\langle x \rangle$  and the amplification factor  $\mathcal{A}$  in all orders of  $k$ . Let  $x_\ell$ ,  $x_m$  and  $x$  be the transverse displacement at the lens, mirror and detector, respectively. We have  $x_\ell/s_i = x_m/l_{\text{im}} = x/l_{\text{id}}$ , where  $s_i$  be the image distance, and  $l_{\text{im}}$  and  $l_{\text{id}}$  are the path lengths between the image and the mirror, and between the image and the detector, respectively. Let the initial variances of  $x_\ell$ ,  $x_m$  and  $x$  be  $a^2$ ,  $a_m^2$  and  $\sigma^2$ . Then we have  $x_\ell/a = x_m/a_m = x/\sigma$  and

$a_m = (1 - \gamma)a + \gamma\sigma$ . It is the most convenient to use  $x_m$  in the calculation because the nontrivial state evolution is caused by the tilted mirror.

The initial state is given by

$$|\Psi_i\rangle = |\Phi_i\rangle|i\rangle, \quad |\Phi_i\rangle = \int dx_m |x_m\rangle \Phi_{m,i}(x_m), \quad (7)$$

where  $\Phi_{m,i}(x_m)$  is the initial wavefunction at the mirror. [The relation to the initial wavefunction  $\Phi_{\ell,i}(x_\ell)$  at the lens is given by  $\Phi_{m,i}(x_m) = \Phi_{\ell,i}(x_\ell) = \Phi_{\ell,i}(s_i x_m / l_{im})$ .] The photons gain a transverse momentum at the mirror, which is described by a unitary operator  $e^{-ikx_m A}$ . The state of the system and the meter becomes

$$|\Psi\rangle = \int dx_m |x_m\rangle \Phi_{m,i}(x_m) e^{-ikx_m A} |i\rangle. \quad (8)$$

With the post-selected state  $|f\rangle$  of the system, the resulting state of the meter reads

$$|\Phi_f\rangle = \langle f | \Psi \rangle = \int dx_m |x_m\rangle \Phi_{m,i}(x_m) \langle f | e^{-ikx_m A} | i \rangle, \quad (9)$$

where the state is not normalized. From  $A^2 = 1$ , we can exactly calculate  $\langle f | e^{-ikx_m A} | i \rangle$  to all orders:

$$\begin{aligned} & \langle f | e^{-ikx_m A} | i \rangle \\ &= \sum_{n=0}^{\infty} \frac{(-ikx_m)^{2n}}{(2n)!} \langle f | i \rangle + \sum_{n=0}^{\infty} \frac{(-ikx_m)^{2n+1}}{(2n+1)!} \langle f | A | i \rangle \\ &= \langle f | i \rangle (\cos kx_m - iA_w \sin kx_m). \end{aligned} \quad (10)$$

From (9) and (10), we have

$$\langle x_m^n \rangle = \frac{\langle \Phi_f | x_m^n | \Phi_f \rangle}{\langle \Phi_f | \Phi_f \rangle} = \frac{\langle x_m^n | \cos kx_m - iA_w \sin kx_m |^2 \rangle_i}{\langle | \cos kx_m - iA_w \sin kx_m |^2 \rangle_i}. \quad (11)$$

Thus we obtain a formula for the expectation value of any moment of  $x$ ,

$$\begin{aligned} \langle x^n \rangle &= \left( \frac{\sigma}{a_m} \right)^n \langle x_m^n \rangle = \left( \frac{\sigma}{a_m} \right)^n \\ &\cdot \frac{\alpha_+ \langle x_m^n \rangle_i + \alpha_- \langle x_m^n \cos 2kx_m \rangle_i + \text{Im} A_w \langle x_m^n \sin 2kx_m \rangle_i}{\alpha_+ + \alpha_- \langle \cos 2kx_m \rangle_i + \text{Im} A_w \langle \sin 2kx_m \rangle_i}, \end{aligned} \quad (12)$$

where  $\alpha_{\pm} := (1 \pm |A_w|^2)/2$ . The expectation value  $\langle x \rangle$  is given by (12) with  $n = 1$ . Then the amplification factor is given by (6). The amplification factor depends on  $k$ , namely, the amplification is nonlinear. We note that we were able to write the result (12) in terms only of  $A_w$ , by virtue of  $A^2 = 1$ ; otherwise the result would depend on  $\langle f | A^k | i \rangle$ .

In the present model, the weak value is pure imaginary,  $A_w = i \cot \frac{\phi}{2}$ . For simplicity, we assume that  $|\Phi_{m,i}(x_m)|^2$

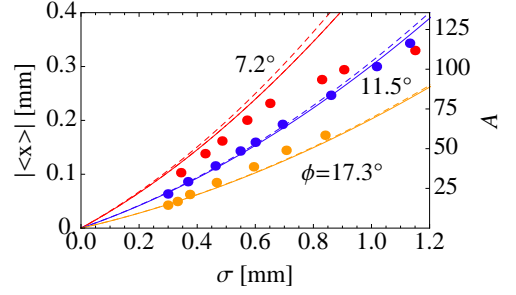


FIG. 2: (Color online) The meter value  $|\langle x \rangle|$  and the amplification factor  $\mathcal{A}$  in the weak measurement as a function of the beam radius  $\sigma$  at the detector, for three values of  $\phi$ . The points are the experimental data from [13]. The solid curve is the theoretical meter value (13) by the nonlinear analysis. The dashed curve is the one (5) by the linear analysis which is proportional to the weak value  $A_w$ .

is an even function. Then from (12), we have a simple expression for the expectation value:

$$\langle x \rangle = \frac{\sigma}{a_m} \frac{\sin \phi \langle x_m \sin 2kx_m \rangle_i}{1 - \cos \phi \langle \cos 2kx_m \rangle_i}. \quad (13)$$

This formula gives the measured value obtained from the weak measurement, for arbitrary  $\Phi_{m,i}(x_m)$ . In the lowest order in  $k$ , (13) yields  $\langle x \rangle = 2ka_m \sigma \cot \frac{\phi}{2}$ , which reproduces the result (5) by Dixon et al. When  $\phi$  varies, one can see that  $|\langle x \rangle|$  attains its maximum

$$|\langle x \rangle|_{\max} = \frac{\sigma}{a_m} \frac{\langle x_m \sin 2kx_m \rangle_i}{\sqrt{1 - \langle \cos 2kx_m \rangle_i^2}}, \quad (14)$$

at  $\cos \phi = \langle \cos 2kx_m \rangle_i$ . In particular, when  $ka_m$  is small (recall that  $\langle x_m^2 \rangle = a_m^2$ ), (14) implies

$$|\langle x \rangle|_{\max} = \sigma + O(ka_m)^2. \quad (15)$$

One can see from (13) and (14) that both the displacement  $|\langle x \rangle|$  and the amplification factor  $\mathcal{A}$  are *bounded* even when the PPS states become orthogonal. In the linear analysis of the weak measurement, the obtained value was proportional to the weak value. This implies that it diverges when  $\phi \rightarrow 0$ , so that one can achieve arbitrarily large amplification factor  $\mathcal{A}$ , though the power of the output beam becomes extremely small. The full order result shows that the measured value of the weak measurement is not proportional to the weak value. The observed value  $\langle x \rangle$  does not diverge but go to zero when  $\phi \rightarrow 0$ . Thus, when the PPS states become orthogonal, the amplification factor  $\mathcal{A}$  is bounded (in fact, vanishes). In other words, one cannot achieve infinite amplification even one compromises on the power of the output beam or the probability of successful post-selection.

To evaluate (13) or (14) explicitly, let us consider the case that the beam is Gaussian,

$$|\Phi_{m,i}(x_m)|^2 = \frac{e^{-x^2/2a_m^2}}{\sqrt{2\pi} a_m}. \quad (16)$$

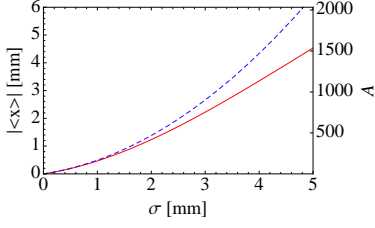


FIG. 3: (Color online) The same as the case  $\phi = 7.2^\circ$  in Figure 2 but with larger scale. The solid (dashed) curve is  $|\langle x \rangle|$  by nonlinear (linear) analysis (13) [(5)].

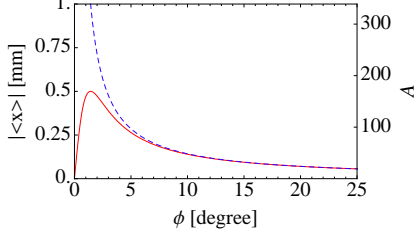


FIG. 4: (Color online) The theoretical meter value  $|\langle x \rangle|$  and the amplification factor  $\mathcal{A}$  in the weak measurement as a function of the phase difference  $\phi$ . The solid curve is from the nonlinear analysis (13). The dashed curve is from the linear analysis (5) which is proportional to the weak value  $A_w$ .

We have  $\langle \cos 2kx_m \rangle = e^{-2(ka_m)^2}$  and  $\langle x_m \sin 2kx_m \rangle = 2ka_m^2 e^{-2(ka_m)^2}$  so that

$$\langle x \rangle = \frac{2ka_m \sigma \sin \phi}{e^{2(ka_m)^2} - \cos \phi}, \quad (17)$$

$$|\langle x \rangle|_{\max} = \frac{2ka_m \sigma}{\sqrt{e^{4(ka_m)^2} - 1}} \text{ at } \cos \phi = e^{-2(ka_m)^2}. \quad (18)$$

Recall that  $a_m = (1 - \gamma)a + \gamma\sigma$ .

Figure 2 shows the theoretically derived meter value  $|\langle x \rangle|$  in the weak measurement as a function of the beam radius  $\sigma$  at the detector. The parameters are chosen the same as Figure 2 of the experiment carried out in [13],  $a = 640 \mu\text{m}$ ,  $\gamma = .296$ ,  $k = 20.8 \text{ m}^{-1}$ . For the larger phase differences  $\phi$ , the value  $|\langle x \rangle|$  in the nonlinear analysis (13) essentially has no difference to that in the linear analysis (5). For the smallest phase difference  $\phi$ , the nonlinear value gives slightly better fit to the experimental data but still does not fit well to them. (The authors of [13] suggested that the difference may be due to the effect of stray light.) As a reference, Figure 3 shows the same theoretical meter values in a larger scale (though it may not be realistic in the performed experiment). It can be seen that, in the setup in [13], the difference between linear and nonlinear results would be significant when  $\sigma$  is larger than 1-2 mm when the amplification is larger than several hundred.

Figure 4 shows the theoretically derived meter value  $|\langle x \rangle|$  in the weak measurement, as a function of the phase

difference  $\phi$ . We chose a typical value of  $\sigma = 500 \mu\text{m}$ . One can verify that when the PPS states become orthogonal,  $\phi \rightarrow 0$ , the  $\langle x \rangle$  in the linear analysis (5) diverges while that in the nonlinear analysis (13) converges to zero. One can also verify  $|\langle x \rangle|_{\max} \lesssim \sigma$  implied by (15) or (18). One finds that the smallest phase difference  $\phi = 7.2^\circ$  chosen in the experiment in [13] is on the border of valid linear analysis, beyond which one needs nonlinear analysis to measure the original tilt angle of the mirror.

*Conclusion.*— We have performed an all-order analysis for the amplifier by Sagnac interferometer. In the linear analysis, the measured meter value  $\langle x \rangle$  was proportional to the weak value (1) and hence diverges in the limit  $\phi \rightarrow 0$  when the pre- and post-selected states be orthogonal. In the nonlinear analysis, the measured value is no longer proportional to the weak value and vanishes in the limit  $\phi \rightarrow 0$ . It is suggested that nonlinear analysis of the weak measurement is necessary if we improve the amplifier by the Sagnac interferometer. In a realistic setting, we have seen that this happens when the amplification factor  $\mathcal{A}$  goes beyond a typical value of roughly 100, and that the maximum possible value of  $\mathcal{A}$  resulting from the nonlinear analysis is several hundred.

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